$\boxed{1}$ Find $V(\vec{r})$ by integrating $d\vec{l} \cdot \vec{E}(\vec{r})$ for a sphere w/ radius R and constant charge density $\rho(\vec{r}) = \rho_0$.

- We used Gauss's Law to find $\vec{E}(\vec{r})$ for this distribution of charge. Since it is spherically symmetric we put the origin @ the center of the sphere, and $\vec{E}(\vec{r})$ is:

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0} \dfrac{\hat{r}}{r^2}, & r \geq R \\ \dfrac{Q}{4\pi\varepsilon_0} \dfrac{r}{R^3} \hat{r}, & r \leq R \end{cases} \qquad \text{w/ } Q = \frac{4}{3}\pi R^3 \rho_0$$

- To find $V(\vec{r})$ we need to pick a reference point $\vec{r}_{RP}$ & then integrate

$$V(\vec{r}) = -\int_{\vec{r}_{RP}}^{\vec{r}} d\vec{l}' \cdot \vec{E}(\vec{r}')$$

along <u>any</u> path from $\vec{r}_{RP}$ to $\vec{r}$.

- Where do we put the reference point? The further we get from the sphere the less we notice its electric field. If we were ∞-far away we wouldn't notice it @ all. So let's say $\vec{r}_{RP}$ is any point very, very far away — so much larger than R that we treat $r_{RP} \sim \infty$.

- Now what path do we follow?

- Moving along a path from $\vec{r}_{RP}$ to $\vec{r}$ involves infinitesimal displacements

$$d\vec{\ell}' = dr'\hat{r} + r'd\theta'\hat{\theta} + r'\sin\theta' d\phi' \hat{\phi}$$

- The details (how dr', $d\theta'$, & $d\phi'$ change relative to each other to keep you on the path) depends on which path you follow. But since $\vec{E} \propto \hat{r}$, all this integral cares about is the dr' part:

$$d\vec{\ell}' \cdot \vec{E}(r') = E(r')\, dr'$$

- Since $\vec{E}$ depends only on r (dist. from $r=0$) and the integral of $d\vec{\ell}' \cdot \vec{E}(r')$ doesn't care about θ' or ϕ', the potential @ $\vec{r}$ depends only on $|\vec{r}| = r$:

$$V(r) = -\int_{r_{RP}}^{r} dr' E(r') \quad \text{w/} \quad E(r') = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0} \dfrac{1}{r'^2}, & r' \geq R \\ \dfrac{Q}{4\pi\varepsilon_0} \dfrac{r'}{R^3}, & r' \leq R \end{cases}$$

- So @ a point <u>outside</u> the sphere

$$V(r>R) = -\frac{Q}{4\pi\varepsilon_0}\int_{r_{RP}}^{r} dr' \frac{1}{r'^2} = -\frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r'}\Big|_{r_{RP}}^{r}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_{RP}}\right)$$

We decided to put $\vec{r}_{RP}$ very far away, $r_{RP} \sim \infty$, so:

$$V(r>R) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

Same as for pt. charge Q @ the origin. Of course! $\vec{E}(\vec{r})$ looks like pt. charge for $r \geq R$, so potential does as well.

- For points inside the sphere, $r < R$:

$$V(r<R) = -\int_{r_{RP}}^{r} dr'\, E(r') = -\int_{R}^{r} dr'\, E(r'<R) - \int_{r_{RP}}^{R} dr'\, E(r'>R)$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_R^r dr'\, \frac{r'}{R^3} - \frac{Q}{4\pi\varepsilon_0}\int_{r_{RP}}^{R} dr'\, \frac{1}{r'^2}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\frac{1}{R^3}\left(\frac{1}{2}r'^2\Big|_R^r\right) - \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r'}\Big|_{r_{RP}}^{R}\right)$$

$$= -\frac{Q}{4\pi\varepsilon_0}\frac{1}{R^3}\left(\frac{1}{2}r^2 - \frac{1}{2}R^2\right) + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R} - \cancelto{\sim 0}{\frac{1}{r_{RP}}}\right)$$

$$\Rightarrow V(r<R) = \frac{Q}{8\pi\varepsilon_0}\times\left(\frac{3}{R} - \frac{r^2}{R^3}\right)$$

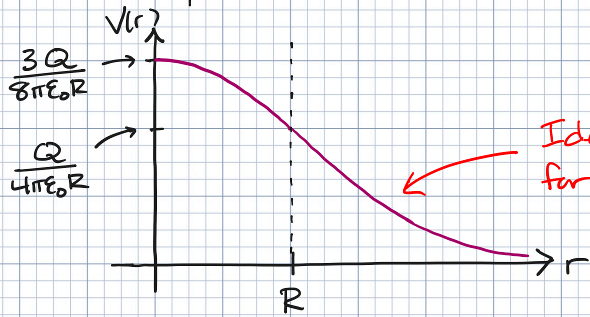
Check for yourself that $-\vec{\nabla}V = -\frac{\partial V}{\partial r}\hat{r}$ gives $\vec{E}$ inside the sphere.

- And that's it. Because of spherical symmetry the electrostatic potential @ a point $\vec{r}$ only depends on that point's distance from the center of the charge distribution. It behaves differently for $r>R$ & $r<R$ because $\vec{E}$ behaves differently in those regions.

$$V(\vec{r}) = \begin{cases} \dfrac{Q}{8\pi\varepsilon_0}\times\left(\dfrac{3}{R} - \dfrac{r^2}{R^3}\right)\hat{r}, & r \le R \\[2ex] \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{r}, & r \le R \end{cases}$$

Notice that both expressions agree @ $r=R$. Later on we'll see that $\vec{E}$ can have 'jumps', but V is <u>always</u> continuous.

- If we plot this it looks like:

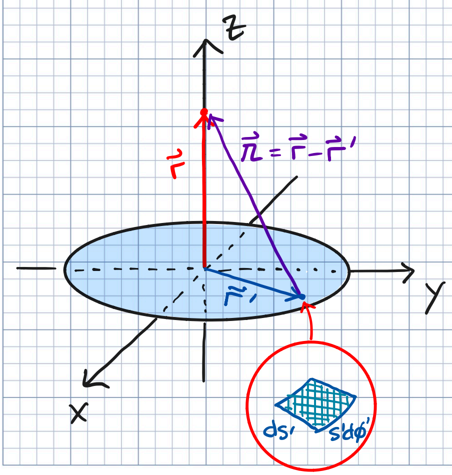


Identical to a point charge for $r \geq R$

$\boxed{2}$ Our first example of a Coulomb integral was finding $\vec{E}$ @ a point directly above the center of a disk w/ constant surface charge density σ_0. Find V @ that point by evaluating:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S da' \frac{\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

- As before, use Cylindrical Polar Coords w/ the origin @ the center of the disk:



DISK: $0 \leq s' \leq R,\ 0 \leq \phi' < 2\pi,\ z' = 0$

$\rightarrow \vec{r}\,' = s'\hat{s} + 0\hat{z}$

PT. ABOVE CENTER: $\vec{r} = z\hat{z}$

$\vec{r} - \vec{r}\,' = -s'\hat{s} + z\hat{z}$

$\rightarrow |\vec{r} - \vec{r}\,'| = \sqrt{s'^2 + z^2}$

CHARGE @ $\vec{r}\,'$: $dq(\vec{r}\,') = \sigma_0\, da'$

$= \sigma_0\, s'd\phi' ds'$

- So we visit every point on the disk & add up all the contributions to $V(0,0,z)$:

$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi' \int_0^R ds'\, s' \frac{\sigma_0}{\sqrt{s'^2+z^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} 2\pi\sigma_0 \int_0^R ds' \frac{s'}{\sqrt{s'^2+z^2}}$$

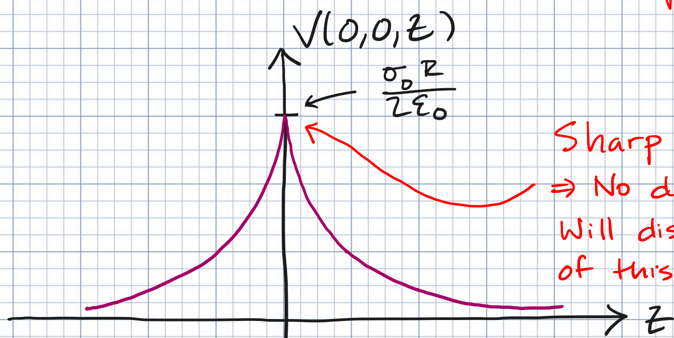
$u = s'^2 + z^2 \qquad s'=0 \rightarrow u = z^2$

$du = 2s'ds' \qquad s'=R \rightarrow u = R^2+z^2$

$$= \frac{\sigma_0}{2\varepsilon_0}\int_{z^2}^{R^2+z^2} du\, \frac{1}{2}\frac{1}{\sqrt{u}} = \frac{\sigma_0}{2\varepsilon_0} \sqrt{u}\,\Big|_{z^2}^{R^2+z^2}$$

$$\rightarrow V(0,0,z) = \frac{\sigma_0}{2\varepsilon_0} \times \left(\sqrt{R^2+z^2} - \sqrt{z^2}\right)$$

CAREFUL! This is $\sqrt{z^2} = |z|$, not $\sqrt{z^2} = z$.



Sharp point @ $z=0$
$\Rightarrow$ No derivative there.
Will discuss meaning of this next class.

- Remember: We set up the integral to find V @ the point directly above the center of the disk. We're free to change z & move up/down the axis, but the potential changes if we move off the z-axis.

- Does this agree w/ our result for $\vec{E}(0,0,z)$ from class?

$$E_z(0,0,z) = -\frac{d}{dz} V(0,0,z)$$

$$= -\frac{\sigma}{2\varepsilon_0} \times \left(\frac{z}{\sqrt{R^2+z^2}} - \frac{z}{\sqrt{z^2}} \right)$$

$$= \frac{\sigma}{2\varepsilon_0} \times \left(\frac{z}{\sqrt{z^2}} - \frac{z}{\sqrt{R^2+z^2}} \right)$$

- What if $z \ll R$? In that case we found $\vec{E} = \pm \frac{\sigma_0}{2\varepsilon_0} \hat{z}$, depending on whether $z > 0$ or $z < 0$.

$$V(0,0,z \ll R) = \frac{\sigma_0}{2\varepsilon_0} \left(R \underbrace{\sqrt{1 + \left(\frac{z^2}{R^2}\right)}}_{1 + \frac{1}{2}\left(\frac{z}{R}\right)^2 - \frac{1}{8}\left(\frac{z}{R}\right)^4 + \ldots} - \sqrt{z^2} \right)$$

$$\simeq \frac{\sigma_0}{2\varepsilon_0} \times \left(R - \sqrt{z^2} + \ldots \right)$$

Very small if $z \ll R$

$$\rightarrow V(0,0,z \ll R) \simeq \frac{\sigma_0}{2\varepsilon_0} R - \frac{\sigma_0}{2\varepsilon_0} \sqrt{z^2}$$

$$E_z(0,0,z \ll R) = -\frac{d}{dz} V(0,0,z \ll R)$$

$$\simeq -\cancelto{0}{\frac{d}{dz}\left(\frac{\sigma_0}{2\varepsilon_0} R\right)} + \frac{d}{dz}\left(\frac{\sigma_0}{2\varepsilon_0} \sqrt{z^2}\right)$$

$$\simeq \frac{\sigma_0}{2\varepsilon_0} \frac{z}{\sqrt{z^2}}$$

1 if $z > 0$,
-1 if $z < 0$